Renormalization of O(N) model in 1/N expansion in auxiliary field formalism

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Abstract

We study the renormalization of the O(N) model using the auxiliary field formalism (Hubbard-Stratonovich transformation) in the 1/N expansion at finite temperature. We provide the general strategy of renormalization for arbitrary order, and make calculation up to next-to-leading order. We show that renormalization is possible for any values of the condensates, prove the temperature independence of the counterterms and determine the cutoff dependence of the first nontrivial counterterm parts.

1   Introduction

Renormalization of approximations in quantum field theories which involve resummation represents a complicated, much studied subject. Depending on the system, various techniques are developed. Renormalization of mass and coupling constant resummation was studied in [1],[2], in the framework of 2PPI-resummation in [3]. Renormalization of Hartree-Fock resummation is performed in [4],[5],[6],[7]. The task of the renormalization of 2PI resummation was solved by [8],[9],[10], [11],[12], its generalization to n-point irreducible resummations is discussed in [13]. Application of this method to scalar models [14],[15],[16] and gauge theories [17],[18] was done. The problem of how the 2PI counterterms can be constructed is discussed in [19]. In [20] a renormalization scheme method was proposed to accomplish 2PI resummation and renormalization in a single step. Renormalization of Schwinger-Dyson equations and its relation to the 2PI approach, was discussed in [21],[22].

Renormalization of O(N) model beyond the by-now textbook case leading order [23] in $1/N$ expansion was studied recently in [24],[25], using 2PI techniques in [26]. In [25] the authors presented a technique, with help of which they could provide a renormalized, background dependent free energy up to next-to-leading order. Although they used auxiliary field formalism (Hubbard-Stratonovich transformation), after a point they switched to a simpler background, eliminating the auxiliary field expectation value using its equation of motion (EoM). It is not clear, whether there exists a consistent free energy also for the auxiliary field background. The problem is that the auxiliary field represents a composite operator of the original fields, and so it may show unusual properties [25],[27].

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The primary goal of this paper is to propose a method to consistently determine the auxiliary and scalar field dependent renormalized free energy. It is important to define in a physically sensible way the pressure (minus the saddle point value of the free energy density) and also the different n-point functions at zero external momentum (from the derivatives of the free energy near the saddle point). For a consistent renormalization one has to know the (temperature and condensate-independent) values of the counterterms in a given regularization. Here we use momentum cutoff regularization, and we shall give the leading order values of the counterterms.

The method proposed here represents an application of the general strategy discussed in [20]. We define a generalized renormalized Lagrangian, where also the counterterms can be momentum dependent, but the bare theory remains intact. The consistency of such renormalization schemes was studied in [20], here we have to check if the asymptotic behavior of the propagator fulfills the generic requirements as derived in [20].

The paper is organized as follows. First the definition of the model is presented, then the renormalized Lagrangian is given in the most general scheme which respects all symmetries of the Lagrangian. Then nontrivial background fields are introduced. For renormalization we first fix the renormalization conditions using the classical Lagrangian, then we determine at leading and next-to leading order the values of the counterterms. We prove that the counterterms are temperature independent, and give formulae for the renormalized free energy. The paper is closed with conclusions.

2 The renormalized Lagrangian

The bare Lagrangian of the system reads:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \bar{\Phi}_i)(\partial^\mu \bar{\Phi}_i) - \frac{\bar{m}^2}{2}\bar{\Phi}_i \bar{\Phi}_i - \frac{\bar{\lambda}}{24N}\left(\bar{\Phi}_i \bar{\Phi}_i\right)^2. \tag{1}$$

For tracing the $1/N$ powers, it is advantageous to split the quartic interaction via the Hubbard-Stratonovich transformation. The above model is equivalent to

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \bar{\Phi}_i)(\partial^\mu \bar{\Phi}_i) - \frac{\bar{m}^2}{2}\bar{\Phi}_i \bar{\Phi}_i - \frac{1}{2}\bar{\chi}^2 - \frac{i\bar{g}}{2\sqrt{N}}\bar{\chi}\bar{\Phi}_i \bar{\Phi}_i, \tag{2}$$

or, in the imaginary time

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \bar{\Phi}_i)(\partial_\mu \bar{\Phi}_i) + \frac{\bar{m}^2}{2}\bar{\Phi}_i \bar{\Phi}_i + \frac{1}{2}\bar{\chi}^2 + \frac{i\bar{g}}{2\sqrt{N}}\bar{\chi}\bar{\Phi}_i \bar{\Phi}_i. \tag{3}$$

where

$$\bar{g} = \sqrt{\frac{\bar{\lambda}}{3}}. \tag{4}$$

In order to be able to do perturbative computations, we should separate in the Lagrangian the renormalized and the counterterm parts. We start with the wave function renormalization:

$$\bar{\Phi} = Z^{1/2}\Phi, \qquad \bar{\chi} = Z_\chi^{1/2}\chi_0. \tag{5}$$

By introducing

$$m_0^2 = Z\bar{m}^2, \qquad g_0 = ZZ_\chi^{1/2}\bar{g}, \tag{6}$$

we can write

$$\mathcal{L}_E = \frac{Z}{2}(\partial_\mu\Phi_i)(\partial_\mu\Phi_i) + \frac{m_0^2}{2}\Phi_i\Phi_i + \frac{Z_\chi}{2}\chi_0^2 + \frac{ig_0}{2\sqrt{N}}\chi_0\Phi_i\Phi_i. \tag{7}$$

As next we should renormalize the operators appearing in the Lagrangian. The basic principle is that all operators which are scalar according to the symmetry group of the Lagrangian and with positive mass dimensions must appear in the renormalized Lagrangian. This means, that we have to include, in addition to the operator set of (7), the operators χ_0 and $(\Phi_i\Phi_i)^2$ – this latter is necessary, since in the actual form of the Lagrangian (7) the $(\Phi_i\Phi_i)^2$ and the χ_0^2 operators are independent, and so they must be renormalized independently. It may turn out, that we do not need counterterm for all these operators, then we can omit them, but, for sake of completeness, we include all possibility here. To allow later a full optimization of the propagator, we introduce a generic kernel and quadratic counterterm for the χ_0 fields. With the notations

$$Z = 1 + \delta Z, \qquad m_0^2 = m^2 + \delta m^2, \qquad Z_\chi = 1 + \delta Z_\chi, \tag{8}$$

the renormalized Lagrangian reads

$$\mathcal{L}_E = \frac{1}{2}\Phi_i\left(-\partial^2 + m^2\right)\Phi_i + \frac{1}{2}\chi_0\, H(i\partial)\, \chi_0 + \frac{ig}{2\sqrt{N}}\chi_0\Phi_i\Phi_i +$$

$$+ \sqrt{N}\, iq\chi_0 + \frac{1}{2}\Phi_i\left(-\delta Z\partial^2 + \delta m^2\right)\Phi_i + \frac{1}{2}\chi_0\,\delta H(i\partial)\,\chi_0 + \frac{i\delta g}{2\sqrt{N}}\chi_0\Phi_i\Phi_i + \frac{\delta\lambda}{24N}(\Phi_i\Phi_i)^2, \tag{9}$$

where the first row is the renormalized, the second is the counterterm part. Some N-power factors are introduced just for convenience. The consistency to the previous form of the Lagrangian requires (changing to Fourier space for easier writing):

$$H(p) + \delta H(p) = Z_\chi. \tag{10}$$

The choice of the kernel is to large extent arbitrary, since – if the above constraint is fulfilled – any choice leaves the bare Lagrangian, and so the physics, untouched. That means that choices which depend on the environment (temperature, singlet backgrounds, etc) are also possible. It is important to note that, if $H(p)$ is chosen to be O(N) singlet, the Lagrangian remains O(N) symmetric, and so in any perturbation theory based on this Lagrangian, the consequences of the symmetry (ie. the Ward identities) remain valid.

A different question is whether a momentum-dependent counterterm can result in, at all, a consistent perturbation theory. This question was addressed in [20]: if the kernel can be power expanded around asymptotic momenta, then the Weinberg theorem is obeyed, so the renormalization program can be fully implemented. We will check later that these conditions are really satisfied.

In this paper we aim to perform expansion with respect to $1/N$ to first order (ie. up to $\mathcal{O}(1/N)$ in the free energy). As it will be later confirmed, we assume the following dependence of the counterterms on $1/N$:

$$q = q_0 + \sum_{n=1}^\infty \frac{1}{N^n} q_n, \qquad \delta f = \delta f_0 + \sum_{n=1}^\infty \frac{1}{N^n}\delta f_n,$$

$$\delta H = \delta H_0 + \sum_{n=1}^\infty \frac{1}{N^n}\delta H_n, \qquad \delta Z_\chi = \delta Z_{\chi,0} + \sum_{n=1}^\infty \frac{1}{N^n}\delta Z_{\chi,n}, \qquad \delta Z = \sum_{n=1}^\infty \frac{1}{N^n}\delta Z_n,$$

$$\delta m^2 = \sum_{n=1}^\infty \frac{1}{N^n}\delta m_n^2, \qquad \delta g = \sum_{n=1}^\infty \frac{1}{N^n}\delta g_n, \qquad \delta\lambda = \sum_{n=1}^\infty \frac{1}{N^n}\delta\lambda_n \tag{11}$$

where δf is the counterterm for the free energy. Note that in certain cases we have zeroth order counterterms as well!

In the perturbative calculation we fix the splitting of the bare parameters into renormalized and counterterm pieces. This separation must be done in a way that the divergences appearing in the expressions of physical quantities cancel at each order. This determines the infinite part of the counterterms. For the finite parts we require that certain physical quantities have pre-defined values (renormalization scheme). For all divergent quantities we should therefore find a physical quantity, which fixes the value of the finite part; in our case we have seven potentially divergent quantities $(q, \delta f, \delta m^2, \delta Z, \delta Z_\chi, \delta g, \delta \lambda)$, so we need seven renormalization conditions. These will be defined through the free energy and its derivatives later.

3   Spontaneous symmetry breaking

In the ground state of the system the bosonic scalar fields can acquire expectation value. In general we can assume

$$\langle \chi_0 \rangle = -i\sqrt{N}\, X, \qquad \langle \Phi_i \rangle = \sqrt{N}\tilde{\Phi}_i. \tag{12}$$

We will allow only those choices for $H(p)$, which depend on the O(N) invariant combination of the background fields, ie. on $\sum_i \tilde{\Phi}_i \tilde{\Phi}_i$. Then the complete Lagrangian will be invariant under a simultaneous O(N) rotation of the background and the fluctuation fields. Then, with an appropriate transformation we can achieve that $\langle \Phi_i \rangle = 0$ for $i = 1, \ldots N-1$, while

$$\langle \Phi_N \rangle = \sqrt{N}\Phi, \qquad \Phi^2 = \sum_i \tilde{\Phi}_i \tilde{\Phi}_i. \tag{13}$$

As a consequence all results can depend only on $\sum_i \tilde{\Phi}_i \tilde{\Phi}_i$, in particular the free energy, too. This ensures that the first and second derivative of the free energy (effective action) for the $k, j < N$ modes can be written as $f(X, \Phi_i) = V(X, \Phi^2)$, and so

$$\frac{\partial f}{\partial \tilde{\Phi}_i} = 2\tilde{\Phi}_i \frac{\partial V}{\partial \Phi}, \qquad \frac{\partial^2 f}{\partial \tilde{\Phi}_i \partial \tilde{\Phi}_j} = 2\delta_{ij} \frac{\partial V}{\partial \Phi} + 4\tilde{\Phi}_j \tilde{\Phi}_k \frac{\partial^2 V}{\partial \Phi^2}. \tag{14}$$

If at the saddle point $\partial_\Phi V = 0$ (we are in the broken phase), the second derivative is a tensor with $N-1$ zero modes. Since the second derivative of the free energy is the inverse propagator at zero momentum, we find $N-1$ zero modes, independently on the value of X. So the Goldstone theorem is satisfied, independently of X.

We introduce fluctuation fields, which already have zero expectation value

$$\chi_0 = -i\sqrt{N}\, X + \chi, \qquad \Phi_N = \sqrt{N}\Phi + \varrho, \qquad \Phi_i = \varphi_i \ (i = 1 \ldots N-1). \tag{15}$$

We write the resulting Lagrangian as

$$\mathcal{L}_E = \mathcal{L}_E^{\text{class}} + \mathcal{L}_E^{\text{class,ct}} + L_E^{\text{lin}} + \mathcal{L}_E^{(2)} + \mathcal{L}_E^{ct,I}, \tag{16}$$

where

$$\mathcal{L}_E^{\text{class}} = N \left[\frac{m^2}{2}\Phi^2 - \frac{H(0)}{2}X^2 + \frac{g}{2}X\Phi^2 \right],$$

$$\mathcal{L}_E^{\text{class,ct}} = N \left[\frac{\delta m^2}{2} \Phi^2 - \frac{\delta H(0)}{2} X^2 + qX + \frac{\delta g}{2} X\Phi^2 + \frac{\delta\lambda}{24}\Phi^4 \right],$$

$$\mathcal{L}_E^{\text{lin}} = \sqrt{N} \left[\left(m_0^2\Phi + gX\Phi + \delta g X\Phi + \frac{\delta\lambda}{6}\Phi^3 \right) \varrho + i \left(-XH(0) + \frac{g}{2}\Phi^2 + q - X\delta H(0) + \frac{\delta g}{2}\Phi^2 \right) \chi \right],$$

$$\mathcal{L}_E^{(2)} = \frac{1}{2}\chi H(i\partial)\chi + \frac{1}{2}\varphi_i \left(-\partial^2 + m^2 + gX \right) \varphi_i + \frac{1}{2} (\varrho \ \chi) \begin{pmatrix} -\partial^2 + m^2 + gX & ig\Phi \\ ig\Phi & H(i\partial) \end{pmatrix} \begin{pmatrix} \varrho \\ \chi \end{pmatrix},$$

$$\mathcal{L}_E^{ct,I} = \frac{ig}{2\sqrt{N}}\chi(\varphi_i\varphi_i + \varrho^2) + \frac{1}{2}\chi\delta H(i\partial)\chi + \frac{1}{2}\varphi_i \left(-\delta Z\partial^2 + \delta m^2 + \delta g X + \frac{\delta\lambda\Phi}{6} \right) \varphi_i +$$

$$+ \frac{1}{2} (\varrho \ \chi) \begin{pmatrix} -\delta Z\partial^2 + \delta m^2 + \delta g X + \frac{\delta\lambda\Phi}{2} & i\delta g\Phi \\ i\delta g\Phi & \delta H(i\partial) \end{pmatrix} \begin{pmatrix} \varrho \\ \chi \end{pmatrix} +$$

$$+ \frac{i\delta g}{2\sqrt{N}}\chi(\varphi_i\varphi_i + \varrho^2) + \frac{\delta\lambda\Phi}{6\sqrt{N}}\varrho(\varphi_i\varphi_i + \varrho^2) + \frac{\delta\lambda}{24N}(\varphi_i\varphi_i + \varrho^2)^2. \tag{17}$$

The φ_i $(i = 1 \ldots N - 1)$ modes will be called pions. Accordingly, we will denote the tree level mass of the pions as:

$$m_\pi^2 = m^2 + gX. \tag{18}$$

Note that the so-defined pion mass depends on the condensate. The quadratic term $\mathcal{L}_E^{(2)}$ provides the propagators, for which the following notations will be used:

$$G_\pi(p) = \frac{1}{p^2 + m_\pi^2}, \qquad \begin{pmatrix} G_{\varrho\varrho} & G_{\varrho\chi} \\ G_{\chi\varrho} & G_{\chi\chi} \end{pmatrix} = \begin{pmatrix} p^2 + m_\pi^2 & ig\Phi \\ ig\Phi & H(p) \end{pmatrix}^{-1}, \tag{19}$$

in particular

$$G_{\chi\chi}(p) = \frac{p^2 + m_\pi^2}{H(p)(p^2 + m_\pi^2) + g^2\Phi^2}. \tag{20}$$

We will compute the constrained free energy or effective potential, which is the 1PI effective action for constant background; with the above separation we can write

$$f = \mathcal{L}_E^{\text{class}} + \frac{N-1}{2} \text{Tr}\log\left(p^2 + m_\pi^2 \right) + \frac{1}{2}\text{Tr}\log\det\begin{pmatrix} p^2 + m_\pi^2 & ig\Phi \\ ig\Phi & H(p) \end{pmatrix} + \frac{1}{\beta V}\left\langle 1 - e^{-S^{ct,I}} \right\rangle_{1PI}. \tag{21}$$

The first term provides the classical result, the rest is due to quantum corrections. The interactions are treated through Taylor expansion:

$$\frac{1}{\beta V}\left\langle 1 - e^{-S^{ct,I}} \right\rangle_{1PI} = \left\langle \mathcal{L}^{ct,I} \right\rangle_{1PI} - \frac{1}{2}\left\langle \int \mathcal{L}^{ct,I}\mathcal{L}^{ct,I} \right\rangle_{1PI} + \ldots. \tag{22}$$

4 Renormalization

After defining the generic framework we determine the necessary counterterms. First we fix the physics by requiring some renormalization conditions. Then order by order compute the necessary diagrams, and require the cancellation of the divergences.

4.1 Renormalization conditions

To fix the renormalization conditions we will require that the radiative corrections do not spoil the robust phenomena of the classical theory. To this end we discuss the classical free energy. Without resummation it reads

$$\frac{f_{cl}}{N} = \frac{m^2}{2}\Phi^2 - \frac{1}{2}X^2 + \frac{g}{2}X\Phi^2. \tag{23}$$

Its saddle point is at position

$$\left.\frac{\partial f_{cl}}{\partial X}\right|_{\mathcal{E}_{ref}} = -X_{\min} + \frac{g}{2}\Phi_{\min}^2 = 0, \qquad \left.\frac{\partial f_{cl}}{\partial \Phi}\right|_{\mathcal{E}_{ref}} = \Phi_{\min}\left(m^2 + gX_{\min}\right) = 0. \tag{24}$$

In the broken phase, where $m^2 < 0$ and $\Phi_{\min} \neq 0$, its solution reads

$$X_{\min} = \frac{-m^2}{g}, \qquad \Phi_{\min}^2 = \frac{-2m^2}{g^2}. \tag{25}$$

The first condition says that the pion mass at the saddle point $m_\pi^2 = m^2 + gX_{\min} = 0$, in accordance with the Goldstone theorem. The rho-mass can be determined from the condition that the ϱ-χ propagator at the saddle point of the free energy has a pole at $p^2 = -m_\varrho^2$. Its position is found by requiring a zero for the determinant of the ϱ-χ kernel at the saddle point:

$$\det\begin{pmatrix} -m_\varrho^2 & ig\Phi \\ ig\Phi & H(-m_\varrho^2) \end{pmatrix} = -m_\varrho^2 H(-m_\varrho^2) + g^2\Phi^2 = 0 \quad \Rightarrow \quad m_\varrho^2 = g^2\Phi_{\min}^2 = -2m^2. \tag{26}$$

Here we used that in the classical case $H(p) = 1$.

In the symmetric phase, where m^2 is positive we obtain $X_{\min} = \Phi_{\min} = 0$, and $m_\pi^2 = m_\varrho^2 = m^2$.

We will require later when we compute radiative corrections, that the position of the saddle point, the value at the saddle point of the free energy and the classical rho mass remain the same. We will also require that the residuum of the pion propagator at $p^2 = 0$ is unity, there is no explicit Φ^4 term, and the coupling of the $X\Phi^2$ term is $g/2$. That is we will require the following renormalization conditions:

$$f_{\mathcal{E}_{ref}} = \frac{-X_{\min}^2}{2}, \qquad \left.\frac{\partial f}{\partial X}\right|_{\mathcal{E}_{ref}} = 0, \qquad \left.\frac{\partial f}{\partial \Phi}\right|_{\mathrm{E}_{ref}} = 0, \qquad \left.\frac{\partial^3 f}{\partial \Phi^2 \partial X}\right|_{\mathrm{E}_{ref}} = g, \qquad \left.\frac{\partial^4 f}{\partial \Phi^4}\right|_{\mathcal{E}_{ref}} = 0,$$

$$\left.\frac{\partial G_\pi^{-1}(p)}{\partial p^2}\right|_{\mathcal{E}_{ref}, p=0} = 1, \qquad H(-m_\varrho^2) = 1, \tag{27}$$

where we denoted

$$\mathcal{E}_{ref} = \{T = 0, (X, \Phi) = (X, \Phi)_{\min}\} \tag{28}$$

as the reference environment.

4.2 Leading order

Here we determine the $\mathcal{O}(N^0)$ counterterms, and give the leading order free energy, as well as the leading order χ propagator.

The free χ propagator is $\mathcal{O}(N^0)$, but there are loop corrections of the same order. Since these loop corrections are not suppressed by the present series expansion parameter, $1/N$, they may appear

in any diagram in arbitrary number. That means that in the naive form at each level of perturbation theory we have infinitely many diagrams. This phenomenon is clearly must be avoided in a well defined perturbation theory, there the radiative corrections must be of lower order in the expansion parameter than the leading term. We can achieve this goal with help of our counterterm $\delta H(p)$: we may tune it in a way that it cancels the $\mathcal{O}(N^0)$ radiative corrections to the χ propagator, or, in other words, we require that the complete self-energy is zero for the χ-field:

$$\Sigma_{\chi\chi}(p,\mathcal{E}) = -\frac{g^2}{2}I(p,\mathcal{E}) - \delta H_0(p) = 0 \qquad \Rightarrow \qquad \delta H_0(p) = -\frac{g^2}{2}I(p,\mathcal{E}), \tag{29}$$

where we explicitly signaled that the computation is performed in a generic $\mathcal{E} = \{T, X, \Phi\}$ environment. The I integral reads as

$$I(p,\mathcal{E}) = \int_q G_\pi(p-q)G_\pi(q)\Big|_{\mathcal{E}}, \tag{30}$$

it is the bubble diagram at environment $\mathcal{E}$. The integration symbol at finite temperature corresponds to

$$\int_q = T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{p}}{(2\pi)^3}, \tag{31}$$

$p_0 = 2\pi nT$, and we apply an UV as well as an IR cutoff to regularize the integrals. We can then calculate the result

$$I(p,\mathcal{E}) = I_0(p) + I_T(p), \tag{32}$$

where the zero temperature part reads

$$I_0(p) = \frac{1}{16\pi^2}\left[\ln\frac{e\Lambda^2}{\bar{m}_\pi^2} + \gamma\ln\frac{\gamma-1}{\gamma+1}\right], \qquad \gamma^2 = 1 + \frac{4\bar{m}_\pi^2}{p^2}, \tag{33}$$

where $\bar{m}_\pi^2 = m_\pi^2 + \Lambda_{IR}^2$ where Λ_{IR} is the IR cutoff. The finite temperature part computed with the Bose-Einstein distribution function $n(\omega)$ is the following:

$$I_T(p_0, p) = \frac{1}{8\pi^2 p}\int_{m_\pi}^{\infty}d\omega\ln\left(\frac{p_0^2 + (p\gamma + 2\omega)^2}{p_0^2 + (p\gamma - 2\omega)^2}\right)n(\omega). \tag{34}$$

We may wonder that the counterterm we choose is environment dependent. However, as we discussed in Section 2, the physics, represented by the bare Lagrangian, is insensitive to the actual choice of the counterterm, as far as the consistency equation (10) is fulfilled. Therefore we must choose:

$$H(p) = 1 + \delta Z_{\chi,0} - \delta H_0(p) = 1 + \delta Z_{\chi,0} + \frac{g^2}{2}I(p,\mathcal{E}) \tag{35}$$

for the free χ-kernel. Here $\delta Z_{\chi,0}$ is part of the bare Lagrangian, and so it cannot be chosen environment dependent. But this is actually unnecessary, since the bubble integral is only overall divergent, and so the divergences do not depend on the IR quantities. Taking into account the renormalization condition (27) we shall choose

$$\delta Z_{\chi,0} = -\frac{g^2}{32\pi^2}\ln\frac{e\Lambda^2}{m_\varrho^2}, \tag{36}$$

then $H(p)$ is a finite quantity with $H(-m_\varrho^2) = 1$.

Some remarks in connection with the leading order result:

- With the procedure above we defined an optimally improved perturbation theory, where all net $\mathcal{O}(N^0)$ radiative corrections are missing (the different contributions cancel each other), and so, as a consequence, at each level only finite number of diagrams are generated. Therefore at higher orders there is no need to repeat this part of the procedure, only ordinary perturbation theory should be used.

- One can realize that at asymptotic momenta the kernel $H(p)$ is logarithmic function of the momentum. This ensures that the renormalizability conditions of [20] are satisfied.

- Later we will need the "double asymptotic" form of H, when both the external p as well as the internal loop momentum (q) go to infinity at the same rate. This form will be relevant for the analysis of overall divergences of higher order diagrams. The result is the same as the $m_\pi = 0$, $T = 0$ case, since these are IR quantities:

$$H_{as}(p) = 1 + \frac{g^2}{32\pi^2} \ln \frac{m_\varrho^2}{p^2} = \frac{g^2}{32\pi^2} \ln \frac{L^2}{p^2}, \qquad L = e^{\frac{16\pi^2}{g^2}} m_\varrho, \tag{37}$$

  L is the position of the Landau pole.

- The original bare coupling of the O(N) model is (cf. (4) and (6)) $\bar{\lambda} = 3g_0^2/(Z_\chi Z^2)$. In the leading order it reads

$$\bar{\lambda} = \frac{3g^2}{1 - \frac{g^2}{32\pi^2} \ln \frac{e\Lambda^2}{m_\varrho^2}} = \frac{\lambda}{1 - \frac{\lambda}{96\pi^2} \ln \frac{e\Lambda^2}{m_\varrho^2}}, \tag{38}$$

  where we defined $\lambda = 3g^2$ as the renormalized quartic coupling. This is the "nonperturbative renormalization" formula for the coupling [24].

In order to be able to fix the other leading order counterterms, we shall consider the free energy. The classical free energy is proportional to N, but also the first quantum correction are of the same order of magnitude. The interactions $(\langle 1 - \exp(-S^{ct,I}) \rangle)$ do not contribute here, just the free pion gas. In $\mathcal{L}^{\text{class,ct}}$ we also have to take into account those terms which contribute at leading order. What we have:

$$\frac{f_0}{N} = \frac{m^2}{2}\Phi^2 - \frac{H(0)}{2}X^2 + \frac{g}{2}X\Phi^2 + q_0 X - \frac{\delta H_0(0)}{2}X^2 + \frac{1}{2}\operatorname{Tr}\log\left(-\partial^2 + m_\pi^2\right) + \delta f_0. \tag{39}$$

We use that $H + \delta H = Z_\chi = 1 + \delta Z_\chi$, and we introduce an environment independent UV regulator to write

$$\frac{f_0}{N} = \frac{m^2}{2}\Phi^2 - \frac{1}{2}X^2 + \frac{g}{2}X\Phi^2 + q_0 X - \frac{\delta Z_{\chi,0}}{2}X^2 + \frac{1}{2}\operatorname{Tr}\log\left(\frac{p^2 + m_\pi^2}{p^2}\right) + \delta f_0. \tag{40}$$

The value of the q_0 and δf_0 counterterms can be determined by the conditions (27). Taking into account the fact that our renormalization conditions were fixed at zero temperature, we find:

$$q_0 = -\frac{g}{2}\int \frac{d^4p}{(2\pi)^4} G_\pi(p)|_{\mathcal{E}_{ref}} + \delta Z_{\chi,0} X_{\min} = \frac{g}{32\pi^2}\left[-\Lambda^2 + m^2 \ln \frac{\Lambda^2}{m_\pi^2} + gX_{\min} \ln \frac{m_\varrho^2}{em_\pi^2}\right],$$

$$\delta f_0 = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log(\frac{p^2 + m_\pi^2}{p^2}) - q_0 X_{\min} + \frac{\delta Z_{\chi,0}}{2}X_{\min}^2 =$$

$$= -\frac{1}{32\pi^2}\left[m^2\Lambda^2 - \frac{m^4}{2}\ln \frac{\Lambda^2}{m_\pi^2} + \frac{g^2 X_{\min}^2}{2}\ln \frac{m_\varrho^2}{em_\pi^2} - \frac{m_\pi^4}{4}\right]. \tag{41}$$

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Note that the divergent parts of q_0 and δf_0 are environment-independent, as it must be. As it can be easily checked, $\delta Z_{\chi,0}$ cancels the divergence coming from the second derivative of the free pion contribution. Then f_0 is a completely finite quantity.

4.3   Next to leading order

The next-to-leading order renormalization means that we must determine the $1/N$ part of the counterterms. Due to the efforts we made at the leading order, this order is the same as any other conventional perturbation theory: since the problematic $\chi\chi$ self energy due to the pion modes is always canceled by the δH counterterm, we will have only a finite number of diagrams to take into account. To determine the counterterms we should consider physical quantities and require that the divergences cancel in the perturbative expressions of these quantities.

First we determine the $\delta Z_1, \delta m_1^2, \delta g_1$ and $\delta\lambda_1$ counterterms from the pion self-energy to the first order:

$$\Sigma_\pi(p) = \frac{g^2}{N}\int_q G_\pi(p-q)G_{\chi\chi}(q) + \delta Z_1 p^2 + \delta m_1^2 + \delta g_1 X + \frac{\delta\lambda_1}{6}\Phi^2 + \frac{\delta\lambda_1}{6}\int_q G_\pi(q). \qquad (42)$$

The first term on the right hand side at first look is only overall divergent, there are no divergent sub-diagrams, since it is only a one-loop diagram. In the overall divergence all momenta go to infinity, and one could think that the divergences cannot depend on the temperature, and the dependence on the field background is polynomial: exactly of the form what are the second to fifth terms in the above expression. But we must not forget, that here the propagator is not the usual quadratic free propagator, and so its temperature dependence can be stronger than an exponential suppression (in fact, in the present case it is $H(p) \sim T^2/p^2$). At the same time the presence of the last term is also embarrassing. It is clear that it is of order $1/N$: although formally it is a next-to-next-to-leading order contribution, but the number of pions $(N-1)$ lifts up this contribution to the next-to-leading order. But what is the role of this diagram?

If we rewrite the $\chi - \chi$ propagator in the original, not-resummed language, the first contribution of (42) corresponds to the diagram of Fig.1/a. But this diagram *has* a subdivergence, which is shown
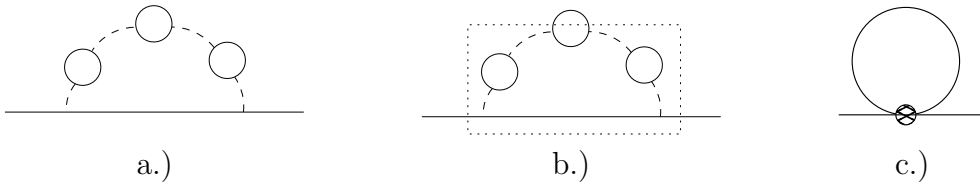

Figure 1: The pion self energy as expanded in the language of the non-resummed $\chi - \chi$ propagator (a.). Its potentially divergent subdiagram boxed by dotted line (b.) The diagram necessary to cancel this subdivergence (c.)

by Fig.1/b. To cancel this subdivergence, we clearly want a counterterm diagram as appears in Fig.1/c, which is exactly the same as the last term in the pion self-energy. The conclusion is that in the original language *the sum of the first and the last term* is overall divergent. This ensures that the $\delta Z_1, \delta m_1^2, \delta g_1$ and $\delta\lambda_1$ counterterms are temperature independent, which is needed for consistency. Their value can be determined using the first term in (42) where *all momenta* in the original,

unresummed representation are asymptotic. In the resummed language it means that we should use the asymptotic pion propagator, and the "double asymptotic" form of $H(p)$ described in (37).

To read off δZ_1 we should consider the p^2 dependence of the overall divergent part of the first term in (42) (the π-χ bubble integral). (37) implies that the χ-χ propagator behaves as $G_{\chi\chi}(p) \sim 1/\ln(p^2)$. So we can write for the overall divergent part of the bubble integral

$$\int d^4 q \frac{1}{(p-q)^2 \ln q^2} \sim \int^\Lambda \frac{dq\, q}{\ln q} \int_0^\pi d\theta \frac{\sin^2 \theta}{1 + (p^2/q^2) + 2p/q \cos \theta} \sim \int^\Lambda \frac{dq\, q}{\ln q} + \text{finite} \times p^4. \qquad (43)$$

Therefore there is no divergent one-loop contribution proportional to p^2. This implies

$$\delta Z_1 = 0. \qquad (44)$$

For the determination of the cutoff dependence of δm_1^2 and δg_1 we include also the pre-factors of (37) and write the overall divergent part of the bubble integral at zero external momentum

$$\delta m_1^2 + \delta g_1 X + \frac{4}{N} \int^{\Lambda^2} \frac{dq\, q^3}{q^2 + m_\pi^2} \frac{1}{\ln L^2/q^2} = \text{finite}. \qquad (45)$$

It means

$$\delta m_{1,div}^2 = \frac{\Lambda^2}{N \ln L/\Lambda} \left[1 - \frac{1}{2 \ln L/\Lambda} + \frac{1}{2 \ln^2 L/\Lambda} \right] + \frac{2m^2}{N} \ln \ln \frac{L^2}{\Lambda^2}, \qquad (46)$$

and

$$\delta g_{1,div} = \frac{2g}{N} \ln \ln \frac{L^2}{\Lambda^2}. \qquad (47)$$

It is interesting to observe that neither δm^2 nor δg is suppressed by powers of the coupling, both are at the same order as m^2 and g, respectively.

To have an expression for the $\delta\lambda_1$ it is simpler to consider another observable, the 4-point function of the scalar fields. It can be expressed at zero external momenta as

$$\Gamma_{iijj,i\neq j}(0) = \frac{2g^4}{N^2} \int_q G_\pi(q) G_{\chi\chi}(q) G_\pi(q) G_{\chi\chi}(q) - \frac{\delta\lambda_1}{N}. \qquad (48)$$

At asymptotic momenta $G_\pi(q) \sim 1/q^2$ and $G_{\chi\chi}(q) \sim 1/\ln(q)$. Therefore the divergence structure is determined by

$$\int \frac{dq\, q^3}{q^4 (\ln q)^2} = \int \frac{d\ln q}{(\ln q)^2}, \qquad (49)$$

which is also UV finite expression. That means that

$$\delta\lambda_1 = 0. \qquad (50)$$

We should remark that the convergence of the one-loop contribution to the wave function renormalization and to the four-point function does not mean that these remain finite at any order. So we still have to keep δZ_2, $\delta\lambda_2$ and all higher counterterms.

For the determination of the counterterms $q_1, \delta Z_{\chi,1}$ and δf_1 we have to write up the effective potential to $\mathcal{O}(N^0)$. We have

$$f_1 = N\left[\frac{\delta m_1^2}{2}\Phi^2 - \frac{Z_{\chi,1}}{2}X^2 + q_1 X + \frac{\delta g_1}{2}X\Phi^2 + \frac{\delta\lambda_1}{24}\Phi^4\right] + \frac{1}{2}\,\text{Tr}\log\left[H(p)(p^2 + m_\pi^2) + g^2\Phi^2\right] +$$

$$+\frac{N}{2}\int_q G_\pi(q)\left[\delta Z_1 p^2 + \delta m_1^2 + \delta g_1 X + \frac{\delta\lambda_1}{6}\Phi^2\right] + \frac{N\delta\lambda_1}{24}\left[\int_q G_\pi(q)\right]^2 + \delta f_1. \tag{51}$$

This is the complete expression, which contains the cancellation of all possible subdivergences. To understand the role of the different terms, the logic is pretty much the same as in case of the pion propagator: the trace-log contribution, although formally only overall divergent, if we expand it with respect to the original, non-resummed modes, we find that it is, in fact, an infinite sum of higher order diagrams, which contain also subdivergences. This is illustrated in Fig.4.3. The relevant diagrams
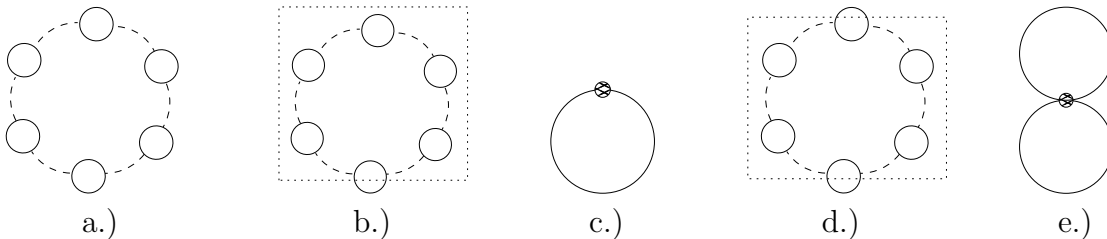

a.)   b.)   c.)   d.)   e.)

Figure 2: The trace-log contribution, expressed with help of the non-resummed propagators (a.). The quadratic subdivergence (b.) and its counterterm (c.). The quartic subdivergence (d.) and its counterterm (e.).

necessary to cancel the subdivergences are the first and second terms in the second line in (51), which are automatically generated in the present framework. In the complete expression no subdivergences survive, only the overall divergent or finite contributions remain. Since in the overall divergent part all momenta go to infinity, there is no way to generate a temperature dependent divergence. Therefore all the necessary counterterms are also temperature independent.

The actual determination of the counterterms is analytically very hard, since it will contain diagrams up to three loop order (each derivatives of $H(p)$ with respect to the background fields raise the diagrammatic loop number). On the other hand if we use numerical techniques, the expression of f_1 contains only one-loop expressions. Since we *know* that there exists a consistent, temperature independent choice for $q_1, \delta Z_{\chi,1}$ and δf_1, we can perform a zero temperature regularized numerical calculation, do the derivations numerically, and determine the correct values for the counterterms.

5 Conclusions

In this paper the renormalization of the O(N) model was discussed. Since the radiative corrections can be of the same order of the expansion parameter (now $1/N$) as the leading order result, a loop-order based perturbation theory is possible only if we include the leading order quantum effects into the quadratic Lagrangian. In 1PI technique it can be achieved by integrating out the pionic modes [25], in the 2PI case one applies an external propagator field to represent the non-trivial propagation. In the present case we used a non-conventional separation of the bare quadratic Lagrangian into free

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and counterterm parts, where the free propagator contains all the leading order corrections. In order to achieve this we need momentum dependent counterterms, too. Since the so-defined free propagator satisfies the conditions of [20], this perturbation theory can be treated with the normal renormalization techniques.

Once we have included the leading radiative corrections into the free propagator, all the rest is conventional perturbation theory. We have to take care, however, that the pion modes can still lift up certain diagrams to higher level, but now it affects only a finite number of diagrams. If we analyze the resulting perturbation theory in the language of the original fields, we can see, that the unusual terms correspond to resummed subdivergences of the leading order result. In our calculation these terms appear automatically, one does not have to perform a detailed BPHZ renormalization to obtain them.

As a result we established the cutoff dependence of the first nontrivial terms of the counterterms, and gave a recipe how numerically can the rest be computed. We gave also the expression for the renormalized free energy up to next-to-leading order.

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